RANDOMNESS VIA INFINITE COMPUTATION AND EFFECTIVE DESCRIPTIVE SET THEORY

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ABSTRACT. We study randomness beyond Π^1_1-randomness and its Martin-Löf type variant, which was introduced in [HN07] and further studied in [BGM17]. Here we focus on a class strictly between Π^1_1 and Σ^1_2 that is given by the infinite time Turing machines (ITTMs) introduced by Hamkins and Kidder. The main results show that the randomness notions associated with this class have several desirable properties, which resemble those of classical random notions such as Martin-Löf randomness and randomness notions defined via effective descriptive set theory such as Π^1_1-randomness. For instance, mutual randoms do not share information and a version of van Lambalgen's theorem holds.

Towards these results, we prove the following analogue to a theorem of Sacks. If a real is infinite time Turing computable relative to all reals in some given set of reals with positive Lebesgue measure, then it is already infinite time Turing computable. As a technical tool towards this result, we prove facts of independent interest about random forcing over increasing unions of admissible sets, which allow efficient proofs of some classical results about hyperarithmetic sets.

CONTENTS

1. INTRODUCTION

Algorithmic randomness studies formal notions that express the intuitive concept of an *arbitrary* or *random* infinite bit sequence with respect to Turing programs. The most prominent such notion is *Martin-Löf randomness* (ML). A real number, i.e. a sequence of length the natural numbers with values 0 and 1, is ML-random if and only if it is not

The second author was partially supported by DFG-grant LU2020/1-1 during the preparation of this paper.

contained in a set of Lebesgue measure 0 that can be effectively approximated by a Turing machine in a precise sense. We refer the reader to comprehensive treatments of this topic in [DH10, Nie09].

Martin-Löf already suggested that the classical notions of randomness are too weak. Moreover, Turing computability is relatively weak in comparison with notions in descriptive set theory. Therefore higher notions of randomness have been considered, for instance, computably enumerable sets are replaced with Π^1_1 sets (see [HN07, BGM17]). These notions were recently studied in [BGM17], and in particular the authors defined a continuous relativization which allowed them to prove a variant of van Lambalgen's theorem for Π^1_1-ML-randomness. We will use this and the Martin-Löf variant of ITTM-random reals in Section 4.3.

There are various desirable properties for a notion of randomness, which many of the formal notions possess, and which can serve as criteria for the evaluation of such a notion. For instance, different approaches to the notion of randomness, such as not having effective rare properties, being incompressible or being unpredictable are often equivalent. Van Lambalgen's theorem states that each half of a random sequence is random with respect to the other half. Moreover, there is often a universal test. For instance ML-randomness and its Π^1_1-variant (see [HN07] and [BGM17] for the relativization) satisfy these conditions. Some types of random reals are not informative and real numbers that are mutually random do not share any nontrivial information. This does not hold for ML-randomness and its variant at the level of Π^1_1, but it does hold for Π^1_1-randomness and the notion of ITTM-randomness studied in this paper.

Higher randomness studies properties of classical randomness notions for higher variants. Various results can be extended to higher randomness notions, assuming sufficiently large cardinals (see e.g. [CY15b]). However, already at the level of Σ^1_2, many properties of randomness notions are independent [CS17]. Therefore we consider classes strictly between Π^1_1 and Σ^1_2.

The infinite time Turing machines introduced by Hamkins and Kidder (see [HL00]) combine the appeal of machine models with considerable strength. The notions decidable, semi-decidable, computable, writable etc. will refer to these machines. The strength of these machines is strictly above Π^1_1 and therefore, this motivates the consideration of notions of randomness based on ITTMs. This project was started in [CS17] and continued in [Car16, Car17].

We consider the following notions of randomness as analogues to Π^1_1-random, Δ^1_1-random and Π^1_1-ML-random reals.

- ITTM-random: avoids every semidecidable null set,
- ITTM-decidable random: avoids every decidable null set,
- $\mathrm{ITTM}_{\mathrm{ML}}$-random: like ML-randomness, but via ITTMs instead of Turing machines.

With respect to the above criteria, they perform differently. As we show below, all notions satisfy van Lambalgen's theorem. We will see that there is a universal test for ITTM-randomness and $\mathrm{ITTM}_{\mathrm{ML}}$-randomness, but not for ITTM-decidable randomness, and we will relate these notions to randomness over initial segments of the constructible hierarchy. A new phenomenon for ITTMs compared to the computable setting is the existence of *lost melodies*, i.e. non-computable recognizable sets (see [HL00]). We will see that lost melodies are not computable from any ITTM-random real. Moreover, we observe that as in [HN07], $\mathrm{ITTM}_{\mathrm{ML}}$-randomness is equivalent to a notion of incompressibility of the finite initial segments of the string.

The first main result is an analogue to a result of Sacks [DH10, Corollary 11.7.2]: computability relative to all elements of a set of positive Lebesgue measure implies computability (asked in [CS17, Section 3]). This result is used in several proofs below.

Theorem 1.1. (Theorem 3.12) Suppose that A is a subset of the Cantor space 2^ω with $\mu(A) > 0$ and a real x is ITTM-computable from all elements of A. Then x is ITTM-computable.

The proof rests on phenonema for infinite time computations that have no analogue in the context of Turing computability, in particular the difference between writable, eventually writable and accidentally writable reals (see Definition 3.1 or [Wel09]).

We state some other main results. We obtain a variant for the stronger *hypermachines* with Σ_n-limit rules [FW11] in Theorem 3.14. We prove a variant of the previous theorem for recognizable sets.[1] Thus we answer several questions posed in [Car17, Section 5] and [Car16, Section 6].

Theorem 1.2. (Theorem 3.16) Suppose that A is a subset of the Cantor space 2^ω with $\mu(A) > 0$ and a real x is ITTM-recognizable from all elements of A. Then x is ITTM-recognizable.

The next result, which is joint with Philip Welch, characterizes ITTM-randomness by the values of an ordinal Σ that is associated to ITTM-computations, the supremum of the ordinals coded by accidentally writable reals, i.e. reals that can be written on the tape at some time in some computation.

Theorem 1.3. (Theorem 4.5) The following conditions are equivalent for a real x.

(a) x is ITTM-random.
(b) x is random over L_Σ and $(\lambda^x, \zeta^x, \Sigma^x) = (\lambda, \zeta, \Sigma)$.
(c) x is random over L_{λ^x}.

The following is a desirable property of randomness that holds for Π^1_1-randomness, but not for Martin-Löf randomness. The property states that mutual randoms do not share non-computable information. Here, two reals are considered random if their join is random.

Theorem 1.4. (Theorem 4.6) If x is computable from both y and z, and y and z are mutual ITTM-randoms, then x is computable.

We further analyze a decidable variant of ITTM-randomness that is analogous to Δ^1_1-randomness. We characterize this notion in Theorem 4.8 and prove an analogue to Theorem 4.6 and to van Lambalgen's theorem for this variant.

All results in this paper, except for the Martin-Löf variant in Section 4.3, work for Cohen reals instead of random reals, often with much simpler proofs, which we do not state explicitly.

The main tool is a variant of random forcing suitable for models of weak set theories such as Kripke-Platek set theory. Previously, some results were formulated for the ideal of meager sets instead of the ideal of measure null sets, since the proofs use Cohen forcing and this is a set forcing in such models. Random forcing, on the other hand, is a class forcing in this situation and it is worthwhile to note that random generic is not equivalent to random over these models (see [Yu11, Remark after Theorem 6.6]). These difficulties are overcome through an alternative definition of the forcing relation, which we call the *quasi-forcing relation*.

As a by-product, the analysis of random forcing allows some more efficient proofs of classical results of higher recursion theory, such as Sacks' theorem that $\{x \mid \omega_1^x > \omega_1^{\mathrm{ck}}\}$ is a null set.

We assume some familiarity with infinite time Turing machines (see [HL00]), randomness (see [Nie09]) and admissible sets (see [Bar75]). Moreover, we frequently use the

[1]An element x of ${}^\omega 2$ is ITTM-recognizable if $\{x\}$ is ITTM-decidable (see Definition 3.15).

Gandy-Spector theorem to represent Π^1_1 sets (see [Hjo10, Theorem 5.5]). In Section 4.3 we will further refer to several proofs in [HN07, Section 3] and [BGM17, Section 3].

The paper is structured as follows. In Section 2, we discuss random forcing over admissible sets and limits of admissible sets. In Section 3, we prove results about infinite time Turing machines and computations from non-null sets. This includes the main theorem. In Section 4, we use the previous results to prove desirable properties of randomness notions.

We would like to thank Laurent Bienvenu for allowing us to include joint results with the first author on ITTM-genericity in Section 4.2. Moreover, we would like to thank Andre Nies, Philip Welch and Liang Yu for discussions related to the topic of this paper and the referee for comments on a previous version.

2. Random forcing over admissible sets

In this section, we present some results about random forcing over admissible sets and unions of admissible sets that are of independent interest. They are essential for the following proofs. The results simplify the approach to forcing over admissible sets (see [Sac90]) by avoiding a ranked forcing language.

We first fix some (mostly standard) notation. A *real* is a set of natural numbers or an element of the Cantor space 2^ω. The basic open subsets of the Cantor space 2^ω will be denoted by $U_s = \{x \in 2^\omega \mid s \subseteq x\}$ for $s \in 2^{<\omega}$. The Lebesgue measure on 2^ω is the unique Borel measure μ with size $\mu(N_t) = 2^{-|t|}$ for all $t \in 2^{<\omega}$. An *admissible set* is a transitive set which satisfies Kripke-Platek set theory with the axiom of infinity. Moreover, an ordinal α is called admissible if L_α is admissible.

2.1. The quasi-forcing relation.

We work with the version of random forcing that is given by Borel codes p for subsets $[p]$ of 2^ω of positive measure, ordered by inclusion. Here we mean any standard way of coding Borel sets by reals or countable trees. In particular, a Borel code in some L_α codes a set that is Borel from the viewpoint of L_α.

It is worthwhile to note that over any admissible set, the following partial order densely embeds into random forcing. The conditions are perfect subtrees of $2^{<\omega}$, i.e. there are no end nodes and splitting nodes above all nodes. A tree is understood as a code for the set $[T]$ of cofinal branches through T.

The results in this section are needed because random forcing is a class forcing over admissible sets, but not necessarily a set forcing. We work with the following reals instead of random generic reals.

Definition 2.1. Suppose that α is an ordinal and $x \in {}^\omega 2$. Then x is *random* over L_α if $x \in A$ for every Borel set A of measure 1 with a Borel code in L_α.

We further distinguish between the forcing relation for random forcing over an admissible set and the *quasi-forcing relation* that is defined below. In its definition, the statement that a set of conditions is dense is replaced with the condition that the union of the conditions has full measure. Thus the quasi-forcing relation corresponds to the random reals defined in Definition 2.1, which are also called quasi-generics (see [Ike10]), as opposed to random generic reals. We will show that this relation is definable over admissible sets, while we do not know if this holds for the forcing relation.

The following two examples illustrate the difference between sufficiently generic and quasi-generic reals.

In the first example, we note that it is easy to construct for any n, dense subsets D of the random forcing in $L_{\omega_1^{ck}}$ such that the union of the conditions in D has measure strictly below $\frac{1}{n}$. To this end, suppose that $\vec{b} = \langle b_\alpha \mid \alpha < \omega_1^{ck}\rangle$ is an enumeration of all Borel codes b_α in $L_{\omega_1^{ck}}$ for Borel sets B_α with positive measure and $f\colon \omega \to \omega_1^{ck}$ is a partial surjection, such that both are Σ_1-definable over $L_{\omega_1^{\mathrm{ck}}}$. We can then construct a sequence of Borel sets $A_\alpha \subseteq B_\alpha$ with $0 < \mu(A_\alpha) < 2^{-(i+n+1)}$ and Borel codes a_α for these sets, where i is least with $f(i) = \alpha$, and let $D = \{a_\alpha \mid \alpha < \omega_1^{\mathrm{ck}}\}$. Note that the sequence $\vec{a} = \langle a_\alpha \mid \alpha < \omega_1^{\mathrm{ck}}\rangle$ can moreover be chosen to be Σ_1-definable over $L_{\omega_1^{\mathrm{ck}}}$, so that D is a Π^1_1 set by the Gandy-Spector theorem [Hjo10, Theorem 5.5].

The second example is Liang Yu's result that $\omega_1^x > \omega_1^{\mathrm{ck}}$ holds for any sufficiently random generic x over $L_{\omega_1^{\mathrm{ck}}}$. This is implicit in [Yu11, Lemma 6.3] and follows from this result with the additional facts that the collection of Π^1_1-ML random reals is $\mathbf{\Sigma}^0_2$ and every Π^1_1-ML-random is Δ^1_1-random [CY15a, Proposition 14.2.2]. It thus follows from Lemma 2.13 below that no sufficiently random generic over $L_{\omega_1^{\mathrm{ck}}}$ avoids every Π^1_1 null set.

We now define Boolean values for the quasi-forcing relation. An ∞-Borel code is a set of ordinals that codes a set built from basic open subsets of 2^ω and their complements by forming intersections and unions of any ordinal length.[2] We will write $\bigvee_{i\in I} x_i$ for the canonical code for the union of the sets coded by x_i for $i \in I$, and similarly for $\bigwedge_{i\in I} x_i$ and $\neg x$.

Definition 2.2. Suppose that L_α is admissible or an increasing union of admissible sets. We define $[\![\varphi(\sigma_0,\dots,\sigma_n)]\!] = [\![\varphi(\sigma_0,\dots,\sigma_n)]\!]^{L_\alpha}$ by induction in L_α, where $\sigma_0,\dots,\sigma_n \in L_\alpha$ are names for random forcing and $\varphi(x_0,\dots,x_n)$ is a formula.

(a) $[\![\sigma \in \tau]\!] = \bigvee_{(\nu,p)\in\tau} [\![\sigma = \nu]\!] \wedge p$.
(b) $[\![\sigma = \tau]\!] = (\bigwedge_{(\nu,p)\in\sigma}([\![\nu \in \tau]\!] \vee \neg p)) \wedge (\bigwedge_{(\nu,p)\in\tau}([\![\nu \in \sigma]\!] \vee \neg p))$.
(c) $[\![\exists x \in \sigma_0\ \varphi(x,\sigma_0,\dots,\sigma_n)]\!] = \bigvee_{(\nu,p)\in\sigma_0} [\![\varphi(\nu,\sigma_0,\dots,\sigma_n)]\!] \wedge p$.
(d) $[\![\neg\varphi(\sigma_0,\dots,\sigma_n)]\!] = \neg[\![\varphi(\sigma_0,\dots,\sigma_n)]\!]$.
(e) $[\![\exists x\ \varphi(x,\tau)]\!] = \bigvee_{\sigma\in L_\alpha} [\![\varphi(\sigma,\tau)]\!]$.

We will leave out the exponent L_α and will further identify $[\![\varphi(\sigma_0,\dots,\sigma_n)]\!]$ with the subset of ${}^\omega 2$ that it codes. The quasi-forcing relation is defined as follows.

Definition 2.3. Suppose that α is admissible or a limit of admissibles, p a random condition in L_α, $\varphi(x_0,\dots,x_n)$ a formula and $\sigma_0,\dots,\sigma_n$ random names in L_α. We define $p \Vdash^{L_\alpha} \varphi(\sigma_0,\dots,\sigma_n)$ if $\mu([p] \setminus [\![\varphi(\sigma_0,\dots,\sigma_n)]\!]) = 0$.

Lemma 2.4. Suppose that α is admissible or a limit of admissibles. Then the function which associates the Boolean value in L_α to Δ_0-formulas $\varphi(\sigma_0,\dots,\sigma_n)$ and the forcing relation for random forcing are Δ_1-definable over L_α.

Proof. The Boolean values are defined by a Δ_1-recursion and the measure corresponding to a code is definable by a Δ_1-recursion. This implies that the forcing relation is Δ_1-definable. □

Definition 2.5. Suppose that α is an ordinal and $x \in {}^\omega 2$. We define $\sigma^x = \{\nu^x \mid (\nu,p) \in \sigma,\ x \in [p]\}$ for $\sigma \in L_\alpha$ by induction on the rank.

(a) The generic extension of L_α by x is defined as $L_\alpha[x] = \{\sigma^x \mid \sigma \in L_\alpha\}$.
(b) The α-th level of the L-hierarchy built over x, with $L_0^x = \mathrm{tc}(\{x\})$, is denoted by L_α^x.

We will see in Lemmas 2.9 and 2.10 that the sets $L_\alpha[x]$ and L_α^x are equal if x is random over L_α and α is admissible or a limit of admissibles.

The next lemma follows by induction on the ranks of names and length of formulas.

Lemma 2.6. Suppose that L_α is admissible or an increasing union of admissible sets, $\sigma_0,\dots,\sigma_n \in L_\alpha$ are names for random forcing and $\varphi(x_0,\dots,x_n)$ is a formula. Then

$$L_\alpha[x] \vDash \varphi(\sigma_0^x,\dots,\sigma_n^x) \iff x \in [\![\varphi(\sigma_0,\dots,\sigma_n)]\!].$$

[2] These codes should not be confused with Borel codes, which are always reals.

The following is a version of the forcing theorem for the quasi-forcing relation.

Lemma 2.7. Suppose that α is admissible or a limit of admissibles, p is a random condition in L_α and $\varphi(x_0, \ldots, x_n)$ is a formula.

(1) If φ is a Δ_0-formula, then $p \Vdash^{L_\alpha} \varphi(\sigma_0, \ldots, \sigma_n)$ holds if and only if $L_\alpha[x] \vDash \varphi(\sigma_0^x, \ldots, \sigma_n^x)$ holds for all random reals $x \in [p]$ over L_α.
(2) If α is countable in L_β, then $p \Vdash^{L_\beta} \varphi(\sigma_0, \ldots, \sigma_n)$ holds if and only if $L_\alpha[x] \vDash \varphi(\sigma_0^x, \ldots, \sigma_n^x)$ for all random reals $x \in [p]$ over L_β.

Proof. For the first claim, we assume that φ is a Δ_0-formula. If $p \Vdash^{L_\alpha} \varphi(\sigma_0, \ldots, \sigma_n)$, then $\mu([p] \setminus [\![\varphi(\sigma_0, \ldots, \sigma_n)]\!]) = 0$. If $x \in [p]$ is random over L_α, then $x \in [\![\varphi(\sigma_0, \ldots, \sigma_n)]\!]$ and hence $L_\alpha[x] \vDash \varphi(\sigma_0^x, \ldots, \sigma_n^x)$ by Lemma 2.6. On the other hand, if $p \not\Vdash^{L_\alpha} \varphi(\sigma_0, \ldots, \sigma_n)$, then $\mu([p] \setminus [\![\varphi(\sigma_0, \ldots, \sigma_n)]\!]) > 0$. If $x \in [p] \setminus [\![\varphi(\sigma_0, \ldots, \sigma_n)]\!]$ is random over L_α, then $L_\alpha[x] \vDash \neg\varphi(\sigma_0^x, \ldots, \sigma_n^x)$ by Lemma 2.6.

The proof of the second claim is analogous, except that now $[\![\varphi(\sigma_0, \ldots, \sigma_n)]\!]$ has a Borel code in L_β instead of L_α. □

The following is a version of the truth lemma for the quasi-forcing relation.

Lemma 2.8. Suppose that α is admissible or a limit of admissibles and x is random over L_α. Then $L_\alpha[x] \vDash \varphi(\sigma^x)$ holds if and only if there is a random condition p in L_α with $x \in [p]$ and $p \Vdash \varphi(\sigma)$.

Proof. Suppose that $x \in [p]$ and $p \Vdash \varphi(\sigma)$. Then $\mu([p] \setminus [\![\varphi(\sigma)]\!]) = 0$. Since x is random over L_α, we have $x \in [\![\varphi(\sigma)]\!]$. Then $x \in L_\alpha[x] \vDash \varphi(\sigma^x)$ by Lemma 2.6.

Suppose that $L_\alpha[x] \vDash \varphi(\sigma^x)$ holds. Then $x \in [\![\varphi(\sigma)]\!]$ by Lemma 2.6. Since $\mu([\![\varphi(\sigma)]\!])$ is the supremum of $\mu([p])$, where p is a condition in L_α with $[p] \subseteq [\![\varphi(\sigma)]\!]$, and x is random over L_α, there is a condition p in L_α with $x \in [p]$. Since $[p] \subseteq [\![\varphi(\sigma)]\!]$, $p \Vdash^{L_\alpha} \varphi(\sigma)$. □

2.2. **The generic extension.** If α is admissible or a limit of admissibles and x is random over L_α, we show that $L_\alpha[x]$ is equal to L_α^x.

Lemma 2.9. For any ordinal β, name $\sigma \in L_\beta$ and real x, we have $\sigma^x \in L_{\beta+2}^x$.

Proof. For any ordinal β, we define the *β-th approximate evaluation* as the function

$$f_\beta : L_\beta \to L^x$$

which maps (τ, p) to τ^x if $x \in [p]$ and to $\emptyset$ otherwise. Moreover we define the *β-th approximation sequence* G_β by letting $G_\beta(\delta) = f_\delta$ for all $\delta < \beta$. In the following, we will show by a simultaneous induction that both $f_\beta \in L_{\beta+1}^x$ and $G_\beta \in L_{\beta+3}^x$ for all ordinals β.

The claim holds for $f_0 = F_0 = \emptyset$. If $\beta = \delta + 1$, then $f_\delta \in L_{\delta+1}^x = L_\beta^x$ by the inductive hypothesis. We then define f_β over L_β^x by

$$f_\beta(\tau, p) = \{f_\delta(\rho, q) \mid x \in [p] \wedge (\rho, q) \in \tau\}$$

for $(\tau, q) \in L_\beta$ so that $f_\beta \in L_{\beta+1}^x$. We further have $F_\beta = F_\delta \cup \{(\delta, f_\delta)\} \in L_{\beta+3}^x$ as required.

If β is a limit ordinal, we let $F_\beta = \bigcup_{\delta<\beta} F_\delta$. Note that for all $\delta < \beta$, the function F_δ is the unique function with domain δ that satisfies the following conditions in L_β^x: $F_\delta(0) = \emptyset$, F_δ is continuous at all limits and is defined as above for successors. Hence F_β is definable over L_β^x. Since $f_\eta = F_\beta(\eta)$ for all $\eta < \beta$, we can now define f_β over L_β^x as

$$f_\beta(\tau, p) = \{F_\beta(\eta)(\rho, q) \mid x \in [p] \wedge \eta < \beta \wedge (\rho, q) \in \tau\}.$$

Since we assumed that $\sigma \in L_\beta$, its evaluation $\sigma^x = \{f_\beta(\rho, q) \mid (\rho, q) \in \sigma\}$ is definable over $L_{\beta+1}^x$. □

Lemma 2.10. Suppose that α is admissible or a limit of admissibles and x is random over L_α. Then $L_\alpha^x \subseteq L_\alpha[x]$.

Proof. It is sufficient to prove this for the case that α is admissible. We will thus show that there is a sequence $\langle \tau_\gamma, \alpha_\gamma \mid \gamma < \alpha \rangle$ that is Σ_1-definable over L_α such that each τ_γ is a name for L^x_γ; this proves the claim since $L_\gamma[x]$ is transitive. Moreover, the ordinals $\alpha_\gamma < \alpha$ will be chosen such that the sequence $\langle \alpha_\gamma \mid \gamma < \alpha \rangle$ is strictly increasing and τ_γ is uniformly Σ_1-definable over L_{α_γ} for all $\gamma < \alpha$.

We pick Borel codes c_n in L_ω for the sets $\{x \in {}^\omega 2 \mid x(n) = 1\}$ and work with the name $\dot{x} = \{(\check{n}, p_n) \mid n \in \omega\} \in L_{\omega+1}$ for the random real. Moreover φ always denotes formulas in the forcing language with a predicate for $\dot{x}$. Let $\tau_0 \in L_{\omega+\omega}$ be a name for $L_0[\dot{x}] = \emptyset$ (with a predicate $\dot{x}$) and $\alpha_0 = \omega + \omega$. Assuming that τ_γ and α_γ are already constructed, we first choose $\alpha_{\gamma+1}$ as follows. Note that the Σ_1-recursion that defines the Boolean values $[\![\varphi^{\tau_\gamma}(\sigma_0, \ldots, \sigma_n)]\!]$ takes place in $L_{\delta_{\sigma_0,\ldots,\sigma_n}}$ for some $\delta_{\sigma_0,\ldots,\sigma_n} < \alpha$, where $\varphi(x_0, \ldots, x_n)$ is any formula and $\sigma_0, \ldots, \sigma_n \in \mathrm{tc}(\tau_\gamma)$ are names. Since α is admissible, there is a least upper bound $\alpha_{\gamma+1} < \alpha$ of $\alpha_\gamma + 1$ and $\delta_{\sigma_0,\ldots,\sigma_n}$ for all $\sigma_0, \ldots, \sigma_n \in \mathrm{tc}(\tau_\gamma)$. Then the Boolean values $[\![\varphi^{\tau_\gamma}(\sigma_0, ..., \sigma_n)]\!]$ are definable over $L_{\alpha_{\gamma+1}}$ uniformly in φ and $\sigma_0, \ldots, \sigma_n$. We now use this fact to define $\tau_{\gamma+1}$. First let $\tau_\gamma^{\varphi,\vec{\nu}} = \{(\sigma, p) \mid \sigma \in \mathrm{tc}(\tau_\gamma),\ p \in L_{\alpha_{\gamma+1}},\ p \Vdash \varphi^{\tau_\gamma}(\sigma, \vec{\nu}))\}$ for all formulas $\varphi(x_0, \ldots, x_n)$ and $\vec{\nu} = (\nu_0, \ldots, \nu_n)$ with $\nu_0, \ldots, \nu_n \in \mathrm{tc}(\tau_\gamma)$. Moreover let $\tau_{\gamma+1}$ be the set of all pairs $(\tau_\gamma^{\varphi,\vec{\nu}}, 1)$, where $\varphi(x_0, \ldots, x_n)$ is a formula and $\vec{\nu} = (\nu_0, \ldots, \nu_n)$ with $\nu_0, \ldots, \nu_n \in \mathrm{tc}(\tau_\gamma)$. Since $p \Vdash \varphi^{\tau_\gamma}(\sigma, \vec{\nu}))$ is equivalent to $p \leq [\![\varphi^{\tau_\gamma}(\sigma, \vec{\nu}))]\!]$, it follows that $\tau_{\gamma+1}$ is definable over $L_{\alpha_{\gamma+1}}$. Finally let $\tau_\gamma = \bigcup_{\beta<\gamma} \tau_\beta$ and $\alpha_\gamma = \sup_{\beta<\gamma} \alpha_\beta$ for limits $\gamma < \alpha$. It is clear that τ_γ is a name for $L_\gamma[\dot{x}]$ and thus the sequence is as required. □

We now argue that $L_\alpha[x]$ is admissible if α is admissible and x is sufficiently random.

Lemma 2.11. Suppose that α is admissible or a limit of admissibles, and x is random over $L_{\alpha+1}$. Then $L_\alpha[x]$ is admissible or a limit of admissibles, respectively.

Proof. It is sufficient to prove this for the case where α is admissible. Suppose that f is a Σ_1-definable function over $L_\alpha[x]$ that is cofinal in α and has domain $\eta < \alpha$. We will assume that $\eta = \omega$ to simplify the notation. Now suppose that $\dot{x}$ is a name for the random generic and $\varphi(n, y, z)$ is a Σ_1-formula that defines the function f in $L_\alpha[x]$ from a parameter with the name $\dot{z}$. Since f is a function in $L_\alpha[x]$ and x is random over $L_{\alpha+1}$, we have

$$\mu(\bigcap_{n\in\omega} [\![\exists\gamma\ \exists y\ \varphi(n, y, \dot{z}) \wedge y \in L_\gamma[\dot{x}]]\!]) > 0$$

by Lemma 2.6. Let $\epsilon = [\![\forall n\ \exists\gamma\ \exists y\ \varphi(n, y, \dot{z}) \wedge y \in L_\gamma[\dot{x}]]\!] = \bigcap_{n\in\omega} [\![\exists\gamma\ \exists y\ \varphi(n, y, \dot{z}) \wedge y \in L_\gamma[\dot{x}]]\!]$; equality holds by the definition of Boolean values.

Claim 2.12. $\mu([\![\forall n\ \exists\gamma\ \exists y\ \varphi(n, y, \dot{z}) \wedge y \in L_\gamma[\dot{x}]]\!] \setminus [\![\exists\gamma\ \forall n\ (\exists y\ \varphi(n, y, \dot{z}) \wedge y \in L_\gamma[\dot{x}])]\!]) = 0$.

Proof. Suppose that $\delta < \epsilon$ with $\delta \in \mathbb{Q}$. We consider the Σ_1-definable function that maps n to the least $\gamma < \alpha$ with

$$\mu(\bigcap_{i\leq n} [\![\exists y\ \varphi(i, y, \dot{z}) \wedge y \in L_\gamma[\dot{x}]]\!]) \geq \delta$$

and this Σ_1-statement (i.e. the statement that the measure is at least δ) is witnessed in L_γ. Since α is admissible, we obtain some $\gamma < \alpha$ with $\mu(\bigcap_{n\in\omega} [\![\exists y\ \varphi(n, y, \dot{z}) \wedge y \in L_\gamma[\dot{x}]]\!]) \geq \delta$. Using the fact that $[\![\forall n\ \exists y\ \varphi(n, y, \dot{z}) \wedge y \in L_\gamma[\dot{x}]]\!] = \bigcap_{n\in\omega} [\![\exists y\ \varphi(n, y, \dot{z}) \wedge y \in L_\gamma[\dot{x}]]\!]$ by the definition of Boolean values, we have

$$\mu([\![\forall n\ \exists\alpha\ \exists y\ \varphi(n, y, \dot{z}) \wedge y \in L_\alpha]\!] \setminus [\![\exists\gamma\ \forall n\ (\exists y\ \varphi(n, y, \dot{z}) \wedge y \in L_\gamma[\dot{x}])]\!]) \leq \epsilon - \delta$$

and since $\delta < \epsilon$ was an arbitrary rational value, the measure is 0. □

Since the Boolean value in Claim 2.12 is definable over L_α and the random real x over $L_{\alpha+1}$ is an element of the set $[\![\forall n\ \exists\gamma\ \exists y\ \varphi(n, y, \dot{z}) \wedge y \in L_\gamma[\dot{x}]]\!]$, it is necessarily also in

$[\![\exists\gamma\ \forall n\ (\exists y\ \varphi(n,y,\dot{z}) \wedge y \in L_\gamma[\dot{x}])]\!]$. It then follows from Lemma 2.6 that the values of f are bounded by some $\gamma < \alpha$, but this contradicts our assumption. □

As an example for how the previous can be applied to prove known theorems, we consider the following classical result (see [Theorem 9.3.9, Nies]). Note that random over $L_{\omega_1^{\mathrm{ck}}}$ in our notation is equivalent to Δ^1_1-random.

Lemma 2.13. (see [Nie09, Theorem 9.3.9]) A real x is Π^1_1-random if and only if x is Δ^1_1-random and $\omega_1^x = \omega_1^{\mathrm{ck}}$.

Proof. We first claim that $\omega_1^x = \omega_1^{\mathrm{ck}}$ for every Π^1_1-random real. The set of random reals over $L_{\omega_1^{\mathrm{ck}}+1}$ has measure 1, and for these reals x, we have $\omega_1^x = \omega_1^{\mathrm{ck}}$ by Lemma 2.11. Moreover $\omega_1^x > \omega_1^{\mathrm{ck}}$ if and only if there is an admissible ordinal in $L_{\omega_1^x}[x]$, hence the set of these reals is Π^1_1 by the Gandy-Spector theorem [Hjo10, Theorem 5.5]. Thus $\omega_1^x = \omega_1^{\mathrm{ck}}$.

In the other direction, let A denote the largest Π^1_1 null set (see [HN07, Theorem 5.2] and Section 4.1 below). By the Gandy-Spector theorem [Hjo10, Theorem 5.5], there are Δ^1_1 null sets A_α for $\alpha < \omega_1^{\mathrm{ck}}$ with $A \subseteq \{x \mid \omega_1^x > \omega_1^{\mathrm{ck}}\} \cup \bigcup_{\alpha<\omega_1^{\mathrm{ck}}} A_\alpha$. Since A is the largest Π^1_1 null set, equality holds. If x is Δ^1_1-random, then $x \notin A_\alpha$ for all $\alpha < \omega_1^{\mathrm{ck}}$ and if we additionally assume that $\omega_1^x = \omega_1^{\mathrm{ck}}$ then $x \notin A$. □

2.3. **Side-by-side randoms.** Two reals x, y are *side-by-side random* over L_α if $\langle x, y\rangle$ is random over L_α for the Lebesgue measure on $2^\omega \times 2^\omega$. The following Lemma 2.16 is analogous to known results for arbitrary forcings over models of set theory, however the classical proof does not work in our setting.

Lemma 2.14. If x, y are side-by-side random over L_α, then x is random over L_α.

To see this, assume that A is a Borel subset of 2^ω of measure 1 with Borel code in L_α; then $\langle x, y\rangle \in A \times 2^\omega$ and hence $x \in A$. We will further use the following lemma.

Lemma 2.15. Suppose that $\langle A_s \mid s \in 2^{<\omega}\rangle$ is a system of Lebesgue measurable subsets of ${}^\omega 2$ such that $A_t \subseteq A_s$ for all $s \subseteq t$ in $2^{<\omega}$ and $\mu(\bigcap_n A_{x\upharpoonright n}) = 0$ for all $x \in 2^\omega$. Then for every $\epsilon > 0$, there is some n such that for all $s \in 2^n$, we have $\mu(A_s) < \epsilon$.

If the lemma fails, then the tree $T = \{s \in 2^{<\omega} \mid \mu(A_s) \geq \epsilon\}$ is infinite. By König's lemma, T has an infinite branch $x \in 2^\omega$ but then $\mu(\bigcap_n A_{x\upharpoonright n}) \geq \epsilon$, contradicting the assumption.

We can now use the forcing theorem for random forcing over admissible sets L_α to prove an analogue to the fact that the intersection of mutually generic extensions is equal to the ground model.

Lemma 2.16. Suppose that L_α is admissible or an increasing union of admissible sets and that x, y are side-by-side random over L_α. Then $L_\alpha[x] \cap L_\alpha[y] = L_\alpha$.

Proof. Let $\mathbb{P}$ denote the random forcing on 2^ω in L_α and $\mathbb{Q}$ the random forcing on $2^\omega \times 2^\omega$ in L_α. Suppose that $z \in L_\alpha[x] \cap L_\alpha[y]$. Moreover, suppose that $\dot{x}, \dot{y}$ are $\mathbb{P}$-names for z with $\dot{x}^x = z$ and $\dot{y}^y = z$. We can assume that $\dot{x}, \dot{y}$ are $\mathbb{Q}$-names by identifying them with the $\mathbb{Q}$-names induced by $\dot{x}$, $\dot{y}$. Then every Borel subset of 2^ω that occurs in $\dot{x}$ is of the form $A \times 2^\omega$ and every Borel subset of 2^ω occuring in $\dot{y}$ is of the form $2^\omega \times A$.

Claim 2.17. No condition p forces over L_α that $\dot{x} = \dot{y}$.

Proof. If $p \Vdash \dot{x} = \dot{y}$ and $\mu([p]) \geq \epsilon > 0$, then $p \Vdash \bigvee_{s \in {}^n 2} \dot{x}\upharpoonright n = \dot{y}\upharpoonright n = s$ for every n by Lemma 2.7. Let $A_s = [\![\dot{x}\upharpoonright n = s]\!]$ and $B_s = [\![\dot{y}\upharpoonright n = s]\!]$, where n is the length of s. We then have $\mu([p] \setminus \bigcup_{s \in {}^n 2}(A_s \times B_s)) = 0$ by Lemma 2.6. Now there is some n with $\mu(A_s) < \epsilon$ for all $s \in {}^n 2$ by Lemma 2.15. Since $\sum_{s \in {}^n 2} \mu(B_s) = 1$, we have $\sum_{s \in {}^n 2} \mu(A_s)\mu(B_s) < \epsilon$. Moreover, the assumption $p \Vdash \dot{x} = \dot{y}$ implies that $\mu([p] \setminus \bigcup_{s \in {}^n 2} A_s \times B_s) = 0$. Therefore $\mu([p]) \leq \mu(\sum_{s \in {}^n 2} \mu(A_s)\mu(B_s)) < \epsilon$, contradicting the assumption that $\mu([p]) \geq \epsilon$. □

This completes the proof of Lemma 2.16. □

3. Computations from non-null sets

In this section, we prove an analogue to the following result of Sacks: any real that is computable from all elements of a set of positive measure is itself computable. This is essential to analyze randomness notions later.

3.1. **Facts about infinite time Turing machines.** An infinite time Turing machine (ITTM) is a Turing machine that is allowed to run for an arbitrary ordinal time, with the rule of forming the inferior limit in each tape cell in each limit step of the computation and moving into a special limit state. The inputs and outputs of such machines are reals.

We recall some basic facts about these machines (see [HL00, Wel09]). The computable sequences are here called *writable* to distinguish this from the following concepts of computability. These notions from [HL00] are interesting on their own and will be essential in the following proofs via results in [Wel09].

Definition 3.1. (See [HL00])

(a) A real x is *writable* (or *computable*) if and only if there is an ITTM-program P such that P, when run on the empty input, halts with x written on the output tape.
(b) A real x is *eventually writable* if and only if there is an ITTM-program P such that P, when run on the empty input, has from some point of time on x written on the output tape and never changes the content of the output tape from this time on.
(c) A real x is *accidentally writable* if and only if there is an ITTM-program P such that P, when run with empty input, has x written on the output tape at some time (but may overwrite this later on).

We write $P^x \downarrow = i$ if P^x halts with output i. The notation Σ_n will always refer to the standard Levy hierarchy, obtained by counting the number of quantifier changes around a Δ_0 kernel.

The ordinal λ is defined as the supremum of the halting times of ITTM-computations (i.e. the *clockable ordinals*), and equivalently [Wel00, Theorem 1.1] the supremum of the writable ordinals, i.e. the ordinals coded by writable reals. Moreover, ζ is defined as the supremum of the eventually writable ordinals, and Σ is the supremum of the accidentally writable ordinals. The ordinals λ^x, ζ^x and Σ^x are defined relative to an oracle x.

We will use the following theorem by Welch [Wel09, Theorem 1, Corollary 2].

Theorem 3.2. (see [Wel09, Theorem 1, Corollary 2]) Suppose that y is a real. Then $\lambda^y, \zeta^y, \Sigma^y$ have the following properties.

(1) $L_{\lambda^y}[y] \cap 2^\omega$ is the set of writable reals in y.
(2) $L_{\zeta^y}[y] \cap 2^\omega$ is the set of eventually writable reals in y.
(3) $L_{\Sigma^y}[y] \cap 2^\omega$ is the set of accidentally reals in y.

Moreover $(\lambda^y, \zeta^y, \Sigma^y)$ is the lexically minimal triple of ordinals with

$$L_{\lambda^y}[y] \prec_{\Sigma_1} L_{\zeta^y}[y] \prec_{\Sigma_2} L_{\Sigma^y}[y].$$

It is worthwhile to note that the precise definition of the Levy hierarchy is important for the reflection in Theorem 3.2. The characterization of λ, ζ and Σ fails if we allow arbitrary additional bounded quantifiers in the Levy hierarchy, since this variant of Σ_2-formulas allows to express the fact that a set is admissible. However L_ζ is admissible [Wel09, Fact 2.2] while L_Σ is not admissible [Wel09, Lemma 6].

We will also use the following information about λ, ζ and Σ.

Theorem 3.3. (1) If the output of an ITTM-program P stabilizes, then it stabilizes before time ζ.
(2) All non-halting ITTM-computations loop from time Σ on.

(3) λ and ζ are admissible limits of admissible ordinals (and more).
(4) In L_λ every set is countable, and the same holds for L_ζ and L_Σ.
Moreover, all of these statements relativize to oracles.

The proofs can be found in [HL00, Wel09]. We will write $x \leq_w y$, $x \leq_{ew} y$, $x \leq_{aw} y$ to indicate that x is writable, eventually writable or accidentally writable, respectively, in the oracle y. The following equivalence is also discussed in [Wel04, page 12].

Lemma 3.4. The following are equivalent for a subset A of ${}^\omega 2$.
(a) A is ITTM-semidecidable.
(b) There is a Σ_1-formula $\varphi(x)$ such that for all $x \in {}^\omega 2$, $x \in A$ if and only $L_{\lambda^x}[x] \models \varphi(x)$.

Proof. In the forward direction, the Σ_1-formula simply states the existence of a halting computation. In the other direction, we can search for a writable code for an initial segment of $L_{\lambda^x}[x]$ which satisfies $\varphi(x)$, using the fact that every set in $L_{\lambda^x}[x]$ has a writable code in x by Theorem 3.2. □

We call a subset of $2^{<\omega}$ *enumerable* if there is an ITTM listing its elements. It follows from Lemma 3.4 that it is equivalent for a subset A of $2^{<\omega}$ that A is semidecidable, A is enumerable or that A is Σ_1-definable over L_λ.

Note that every ITTM-semidecidable set is absolutely Δ^1_2, i.e. it remains Δ^1_2 with the same definition in any inner model and in any forcing extension. Therefore such sets are Lebesgue measurable and have the property of Baire by [Kan09, Exercise 14.4].

3.2. **Preserving reflection properties by random forcing.** The following reflection argument is an essential step in the proof of the preservation of λ, ζ and Σ with respect to random forcing in Section 3.3 below. We show that for admissibles or limits of admissibles $\alpha < \beta$, the statement $L_\alpha \prec_{\Sigma_n} L_\beta$ is preserved to generic extensions for sufficiently random reals.

Definition 3.5. Suppose that A is a Lebesgue measurable subset of ${}^\omega 2$. An element x of ${}^\omega 2$ is a *(Lebesgue) density point of* A if $\lim_n \frac{\mu(A \cap U_{x\upharpoonright n})}{\mu(U_{x\upharpoonright n})} = 1$. Let $D(A)$ denote the set of density points of A.

We will use the following version of Lebesgue's density theorem.

Theorem 3.6. (Lebesgue, see [AC13, Section 8]) If A is any Lebesgue measurable subset of ${}^\omega 2$, then $\mu(A \triangle D(A)) = 0$.

To prove the preservation of Σ_n-reflection, we will need the following result.

Lemma 3.7. Suppose that α is admissible or a limit of admissible ordinals, $t \in 2^{<\omega}$, $\sigma \in L_\alpha$, $\epsilon \in \mathbb{Q}$, $n \geq 1$ and φ is a formula. The formulas in the following claims have the parameters t, σ and ϵ. Let $m_{\sigma,t} = \mu(\llbracket \varphi(\sigma) \rrbracket \cap U_t)$.
(1) If φ is Σ_n, then
  (a) $m_{\sigma,t} > \epsilon$ is equivalent to a Σ_n-formula.
  (b) $m_{\sigma,t} \leq \epsilon$ is equivalent to a Π_n-formula.
(2) If φ is Π_n, then
  (a) $m_{\sigma,t} < \epsilon$ is equivalent to a Π_n-formula.
  (b) $m_{\sigma,t} \geq \epsilon$ is equivalent to a Σ_n-formula.

Proof. The claim holds for Δ_1-formulas φ, since the function mapping σ to $\llbracket \varphi(\sigma) \rrbracket$ is Δ_1-definable in the parameter σ. Assuming that $\varphi(x, y)$ is a Π_n-formula, we now show the first claim for the formula $\exists x \varphi(x, y)$.

We have $\mu(\llbracket \exists x \varphi(x, y) \rrbracket \cap U_t) > \epsilon$ if and only if there is some k and some $\sigma_0, \dots, \sigma_k$ such that $\mu(\llbracket \bigvee_{i \leq k} \varphi(\sigma_i, \tau) \rrbracket \cap U_t) > \epsilon$. By the Lebesgue density theorem (Theorem 3.6), the last inequality is equivalent to the statement that there is some l, a sequence $t_0, \dots, t_l$ of pairwise incompatible extensions of t and some $\epsilon_0, \dots, \epsilon_l \in \mathbb{Q}$ such that $\epsilon = \sum_{j \leq l} \epsilon_j$ and for all $j \leq l$, there is some $i \leq k$ such that $\mu(\llbracket \varphi(\sigma_i, y) \rrbracket \cap U_{t_i}) > \epsilon_j$. Using a universal Σ_n-formula, we obtain an equivalent Σ_n-statement. Moreover, we have $\mu(\llbracket \exists x \varphi(x,y) \rrbracket \cap U_t) \leq \epsilon$ if and only if for all $\sigma_0, \dots, \sigma_k$, $\mu(\llbracket \bigvee_{i \leq k} \varphi(\sigma_i, \tau) \rrbracket) \leq \epsilon$, and this is a Π_n-statement by argument in the previous case.

The second claim follows by switching to negations. □

We can now show the preservation of the statement $L_\alpha \prec_{\Sigma_n} L_\beta$ for sufficiently random reals.

Lemma 3.8. Suppose that β is admissible or a limit of admissibles, x is random over L_β and $L_\alpha \prec_{\Sigma_n} L_\beta$, where $\alpha < \beta$ and $n \geq 1$. If $n \geq 2$, then we additionally assume that x is random over L_γ for some γ such that β that is countable in L_γ. Then $L_\alpha[x] \prec_{\Sigma_n} L_\beta[x]$.

Proof. We first argue that L_α is admissible. If $z \in L_\alpha$ and $f\colon z \to L_\alpha$ is Σ_1-definable over L_α, then the set L_α witnesses Σ_1-collection for f in L_β. Since $L_\alpha \prec_{\Sigma_1} L_\beta$, it follows that Σ_1-collection for f holds in L_α and hence $f \in L_\alpha$.

To prove Σ_n-reflection, we assume that the statement $\exists u\, \varphi(u, \tau^x)$ holds in $L_\beta[x]$, where $n = m+1$, φ is Π_m and $\tau \in L_\alpha$. Moreover, suppose that σ_0 is a name in L_β with $L_\beta[x] \vDash \varphi(\sigma_0^x, \tau^x)$.

Let $B = \llbracket \varphi(\sigma_0, \tau) \rrbracket$. If $n = 1$, then B has a Borel code in L_β. If $n \geq 2$, then B has a Borel code in L_γ by the assumption that β is countable in L_γ. It thus follows from Lemma 2.7 that $x \in B$ and $\mu(B) > 0$. Let A_l denote the set of $s \in 2^{<\omega}$ such that

$$\frac{\mu(B \cap U_s)}{\mu(U_s)} > 1 - 2^{-l}.$$

In the next proof, by an *antichain* in a subset $A^\star$ of $2^{<\omega}$, we mean a subset of $A^\star$ whose elements are pairwise incomparable. Moreover, it is called *maximal* if it is not properly contained in any antichain in $A^\star$.

In the next claim, we conclude from the Lebesgue density theorem that B is almost covered by the sets U_s for $s \in A_n$.

Claim 3.9. If $A^\star$ is a maximal antichain in A_l, then $\mu(B \cap \bigcup_{s \in A^\star} U_s) = \mu(B)$.

Proof. Assume that the claim fails and thus $\mu(B \setminus \bigcup_{s \in B^\star} U_s) > 0$. Then $B \setminus \bigcup_{s \in A^\star} U_s$ has a density point z by the Lebesgue density theorem (Theorem 3.6). Therefore, there is some k with $\frac{\mu(B \cap U_{z \restriction k})}{\mu(U_{z \restriction k})} > 1 - 2^{-l}$ and thus $z \restriction k \in A_l$, by the definition of A_l. However, $z \restriction k$ is incomparable with all elements of $A^\star$, since $z \notin \bigcup_{s \in A^\star} U_s$. This contradicts the assumption that $A^\star$ is maximal. □

We now choose a maximal antichain $A_l^\star$ in A_l for each l. If $n = 1$, then B has a Borel code in L_β, and since β is admissible or a limit of admissibles, we can choose $A_l^\star$ such that the sequence $\langle A_l^\star \mid l \in \omega \rangle$ is an element of L_β. On the other hand, if $n \geq 2$, then B has a Borel code in $L_{\beta+1} \subseteq L_\gamma$ and hence we can choose $A_l^\star$ such that the sequence $\langle A_l^\star \mid l \in \omega \rangle$ is an element of L_γ.

We aim to reflect the Σ_n-statement $\exists v\, \varphi(v, \tau^x)$ from $L_\beta[x]$ to $L_\alpha[x]$. Since σ_0 and the code for B are not necessarily elements of L_α, we will obtain the required objects in L_α by reflection. To this end, we define a subset C of B in L_β with full measure in B such that reflection will hold for all randoms over L_β in C.

Let $B_\sigma = \llbracket \varphi(\sigma, \tau) \rrbracket$, so that in particular $B_{\sigma_0} = B$. We now consider the formula $\psi_k(s)$ stating that there is some name σ with $\frac{\mu(B_\sigma \cap U_s)}{\mu(U_s)} > 1 - 2^{-k}$. This is a Σ_n-statement by Lemma 3.7.

If $s \in A_l$, then $\psi_l(s)$ holds in L_β. Since $L_\alpha \prec_{\Sigma_n} L_\beta$ by our assumption, this implies that $\psi_l(s)$ holds in L_α. For all $s \in A_l$, let σ_s^l denote the $<_L$-least name in L_α witnessing $\psi_l(s)$;

then $\langle\sigma^l_s \mid l\in\omega\rangle$ is an element of L_β for any $s\in A_l$. We further let $C_l=\bigcup_{s\in A^\star_{2l}} B_{\sigma^{2l}_s}$ and $C=\bigcup_l C_l$. If $n=1$, it follows that the sets C_l have Borel codes in L_β for all $l\in\omega$ and moreover, the set C has a Borel code in L_β. If $n\geq 2$, then the same holds for L_γ.

Claim 3.10. $\mu(B\setminus C)=0$.

Proof. We have $\frac{\mu(B\cap U_s)}{\mu(U_s)}>1-2^{-2l}$ for all $s\in A^\star_{2l}$ by the definition of A_{2l} and $\frac{\mu(B_s\cap U_s)}{\mu(U_s)}>1-2^{-2l}$ for all $s\in A_{2l}$ by the choice of σ^{2l}_s. Hence

$$\frac{\mu(B\cap B_s\cap U_s)}{\mu(B\cap U_s)}\geq\frac{\mu(B\cap B_s\cap U_s)}{\mu(U_s)}>1-2^{-l}$$

for all $s\in A_{2l}$. Moreover,

$$\mu(\bigcup_{s\in A^\star_l}(B\cap U_s))=\mu(B\cap\bigcup_{s\in A^\star_l}U_s)=\mu(B)$$

by Claim 3.9. Since $A^\star_l\subseteq A_l$ is an antichain, the sets $B\cap U_s$ for $s\in A^\star_l$ are pairwise disjoint. Therefore the previous inequality implies that

$$\frac{\mu(B\cap C_l)}{\mu(B)}>1-2^{-l}.$$

Since $C=\bigcup_l C_l$, this implies that $\frac{\mu(B\cap C)}{\mu(B)}=1$ and thus $\mu(B\setminus C)=0$. □

Claim 3.11. $\varphi((\sigma^{2l}_s)^x,\tau^x)$ holds in $L_\alpha[x]$.

Proof. We have $x\in B$ by our assumption. We first assume that $n=1$. Since B and C have Borel codes in L_β, $\mu(B\setminus C)=0$ and x is random over L_β, it follows that $x\in C$. If $n\geq 2$, the same argument works for L_γ. Therefore $x\in C_l$ for some l and thus $x\in B_{\sigma^{2l}_s}=[\![\varphi(\sigma^{2l}_s,\tau)]\!]$ for some $s\in A^\star_{2l}$. Now Lemma 2.7 implies that $\varphi((\sigma^{2l}_s)^x,\tau)$ holds in $L_\alpha[x]$. □

The previous claims show that the statement $\exists u\ \varphi(u,\tau^x)$ reflects to $L_\alpha[x]$. □

The assumptions in Lemma 3.8 for $n=2$ are not optimal for the application to ITTMs below. We will see in Section 4.1 that ITTM-randomness is a sufficient assumption for the applications.

3.3. **Writable reals from non-null sets.** We will prove an analogue to the following theorem for infinite time Turing machines. Let $\leq_{\mathrm{T}}$ denote Turing reducibility.

Theorem 3.12. (Sacks, see [DH10, Corollary 11.7.2]) A real x is computable if and only if $\{y\mid x\leq_{\mathrm{T}} y\}$ has positive Lebesgue measure.

In [CS17], analogues of this theorem for other machines were considered. It was asked if this holds for infinite time Turing machines, and this was only proved for non-meager Borel sets, via Cohen forcing over levels of the constructible hierarchy. With the results in Section 2, we prove this for Lebesgue measure.

Theorem 3.13. (1) A real x is writable if and only if $\mu(\{y:x\leq_w y\})>0$
(2) A real x is eventually writable if and only if $\mu(\{y:x\leq_{ew} y\})>0$
(3) A real x is accidentally writable if and only if $\mu(\{y:x\leq_{aw} y\})>0$

Proof. The forward direction is clear in each case. In the other direction, we only prove the writable case, since the proofs of the remaining cases are analogous.

Let $W_x:=\{y:x\leq_w y\}$ and choose some sufficiently random $r\in W_x$. Since Σ is a limit of admissible ordinals (see [Wel09, Fact 2.5, Lemma 6]), $L_\Sigma[r]=L^r_\Sigma$ by Lemma 2.9 and Lemma 2.10 and $L_\Sigma[r]$ is an increasing union of admissible sets by Lemma 2.11. We choose some sufficiently random $s \in W_x$ over $L_\Sigma[r]$, in particular s is random over $L_{\Sigma+1}$. Since $L_\lambda \prec_{\Sigma_1} L_\zeta \prec_{\Sigma_2} L_\Sigma$, we have

$$L_\lambda[r] \prec_{\Sigma_1} L_\zeta[r] \prec_{\Sigma_2} L_\Sigma[r]$$

by Lemma 3.8, and we obtain the same elementary chain for s. Since $(\lambda^r, \zeta^r, \Sigma^r)$ and $(\lambda^s, \zeta^s, \Sigma^s)$ are lexically minimal and the values do not decrease in the extensions by r and s, this implies $\lambda = \lambda^r = \lambda^s$, $\zeta = \zeta^r = \zeta^s$ and $\Sigma = \Sigma^r = \Sigma^s$.

We can assume that r is random over L_γ and s is random over $L_\gamma[r]$ for some $\gamma > \Sigma$ such that L_γ satisfies a sufficiently strong theory to prove the forcing theorem and facts about random forcing, and such that generics and quasi-generics over L_γ coincide (see [Jec03, Lemma 26.4]). Since the 2-step iteration of random forcing is equivalent to the side-by-side random forcing (see [BJ95, Lemma 3.2.8]), (r, s) is side-by-side random over $L_{\Sigma+1}$.[3]

Since x is writable relative to r and relative to s, $x \in L_\lambda[r] \cap L_\lambda[s] = \lambda$ by Lemma 2.16, therefore x is writable. □

As far as we know, the following class is the largest class between Π^1_1 and Σ^1_2 that has been studied. We write $x \leq_{n-\mathrm{hyp}} y$ if x is computable from y by a Σ_n-hypermachine introduced in [FW11].

Theorem 3.14. For all $n \geq 1$, a real x is writable by a Σ_n-hypermachine if and only if $\mu(\{y : x \leq_{n-\mathrm{hyp}} y\}) > 0$

The proof is analogous to the proof of Theorem 3.13 via the results of [FW11] and the version of Lemma 3.8 for Σ_n-formulas instead of Σ_2-formulas.

3.4. **Recognizable reals from non-null sets.** We will prove an analogous result as in the previous section, where computable reals are replaced with *recognizable reals* from [HL00]. This is an interesting and much weaker alternative notion to computability. The divergence between computability and recognizability is studied in [HL00, CSW].

A real is recognizable if its singleton is decidable. *Lost melodies*, i.e. recognizable non-computable sets, neither appear in Turing computation nor in the hyperarithmetic setting, since every Δ^1_1 singleton is hyperarithmetic.

Definition 3.15. (a) A real x is *recognizable* from a real y if and only if there is an ITTM-program P such that $P(y)$ halts for every real y, and $P(y \oplus z)$ halts with output 1 if and only if $x = z$.

(b) A real x is a *lost melody* if it is recognizable, but not writable.

A simple example for a lost melody is the constructibly least code for a model of $ZF^- + V{=}L$ [HL00]. It was demonstrated in [Car17, Theorem 3.12] that every real that is recognizable from all elements of a non-meager Borel set is itself recognizable. The new observation for the following proof is that one can avoid computing generics by working with the forcing relation. This also leads to a simpler proof in the non-meager case.

Theorem 3.16. Suppose that a real x is recognizable from all elements of A and $\mu(A) > 0$. Then x is recognizable.

Proof. We can assume that there is a single program P which recognizes x from all oracles in A, since the set of oracles which recognize x for a fixed program is absolutely Δ^1_2 and hence Lebesgue measurable (see [Kan09, Exercise 14.4]).

Let D be the set of the conditions p in L_{λ^x} that decide whether x is accepted or rejected by P relative to the random real y over L_{Σ^x+1}, i.e. either P accepts $x \oplus y$ for all random

[3]Alternatively, the proof of the product lemma or the 2-step lemma [Jec03, Lemma 15.9, Theorem 16.2] can be easily adapted to show directly that (r, s) is side-by-side random over $L_{\Sigma+1}$.

reals $y \in [p]$ over L_{Σ^x+1} or P rejects $x \oplus y$ for all such reals. We will use the simplified notation $\bigcup D$ for the set $\bigcup_{p \in D} [p]$.

Claim 3.17. $\mu(A \setminus \bigcup D) = 0$.

Proof. If the conclusion fails, then there is a random real y over L_{Σ^x+1} in $A \setminus \bigcup D$. Since $P^{x \oplus z}$ converges for any $z \in A$, $P^{x \oplus y} \downarrow = i$ for some i. Since $\lambda^{x \oplus y} = \lambda^x$ by Theorem 3.8 and $L_{\lambda^x}[x \oplus y] = L_{\lambda^x}^{x \oplus y}$ by Lemma 2.9 and Lemma 2.10, there is a name $\dot{C}$ in L_{λ^x} and a condition p in L_{λ^x} with $y \in [p]$ that forces $\dot{C}$ to be a computation of P with input $x \oplus y$ and output i. Then $p \in D$ and $y \in \bigcup D$, contradicting the assumption on y. □

By the Lebesgue density theorem, there is an open interval with rational endpoints for which the relative measure of A is strictly larger than $1 - \epsilon$ for some $\epsilon < \frac{1}{3}$. We can assume that this interval is equal to ${}^\omega 2$.

The procedure Q for recognizing x works as follows. Suppose that $\dot{y}$ is a name for the random real over $L_{\Sigma+1}$. Given an oracle z, we enumerate $L_{\lambda^z}[z]$ via a universal ITTM. In parallel, we search for pairs $(p, \dot{C})$ in $L_{\lambda^z}[z]$ such that p is a condition and $\dot{C}$ is a name such that p forces over $L_{\lambda^z}[z]$ that $\dot{C}$ is a computation of P in the oracle $z \oplus \dot{y}$ that halts with output 0 or 1. Note that these are Δ_0 statements and that the forcing relation for such statements is Δ_1 by Lemma 2.6 and hence ITTM-decidable. We keep track of the conditions that force the corresponding computation to halt with output 0 or with output 1 on separate tapes. Moreover, we keep track of the measures u_0 and u_1 of the union of all conditions on the two tapes. Note that the measure of Borel sets can be computed in admissible sets by a Δ_1-recursion and hence it is ITTM-computable. Since $\mu(A) > 1 - \epsilon$ and $\mu(A \setminus \bigcup D) = 0$, eventually $u_0 + u_1 > 1 - \epsilon$. As soon as this happens, we output 1 if $u_0 > 1 - 2\epsilon$ and 0 otherwise. We claim that Q^z outputs 1 if and only if $z = x$.

Claim 3.18. $Q^x \downarrow = 1$.

Proof. The measure of a countable union of sets can be approximated with arbitrary precision by unions of a finite number of sets. Since $\mu(A \setminus \bigcup D) = 0$ and $\mu(A) > 1 - \epsilon$, $\mu(\bigcup D) > 1 - \epsilon$. There are disjoint conditions $p, q \in L_{\lambda^x}[x]$ with $\mu([p] \cup [q]) > 1 - \epsilon$ such that p forces $Q^{x \oplus \dot{y}} \downarrow = 1$, and q forces $P^{x \oplus \dot{y}} \downarrow = 0$. Since $\mu(\bigcup D) > 1 - \epsilon$, $\mu([q]) \leq \epsilon$ and hence $\mu([p]) > 1 - 2\epsilon$. Eventually, such a condition p will be found and hence the procedure halts with output 1. □

Claim 3.19. $Q^z \downarrow = 0$ if $z \neq x$.

Proof. Suppose that the claim fails. Since Q always halts, we have $Q^z \downarrow = 1$. Then there is a condition p with $\mu([p]) > 1 - 2\epsilon$ which forces $P^{z \oplus \dot{y}} \downarrow = 1$. Since $\mu(A) > 1 - \epsilon$ and $\epsilon < \frac{1}{3}$, $\mu(A \cap [p]) > 0$ and hence there is a random y in $A \cap [p]$ over $L_{\lambda^z}[z]$. Since $y \in [p]$, $P^{z \oplus y} \downarrow = 1$. Since $y \in A$ and $z \neq x$, $P^{z \oplus y} \downarrow = 0$. □

This completes the proof of Theorem 3.16. □

The results in Section 2 also imply analogues of Theorem 3.13 and Theorem 3.16 for other notions of computation and recognizability, for instance the infinite time register machines [CFK+10] and a weaker variant [Koe06]. We explore this in further work.

4. Random reals

We introduce natural randomness notions associated with infinite time Turing machines and show that they have various desirable properties.

This is the motivation for the previous results, which we will apply here. The results resemble the hyperarithmetic setting, although some proofs are different. Theorem 4.8 shows a difference to the hyperarithmetic case.

4.1. **ITTM-random reals.** The following is a natural analogue to Π^1_1-randomness.

Definition 4.1. A real x is ITTM-*random* if it is not an element of any ITTM-semidecidable null set. The definition relativizes to reals.

We first note that there is a universal test. This follows from the following lemma as in [HN07, Theorem 5.2].

Lemma 4.2. We can effectively assign to each ITTM-semidecidable set S an ITTM-semidecidable set $\hat{S}$ with $\mu(\hat{S}) = 0$, and $\hat{S} = S$ if $\lambda(S) = 0$.

Proof. Suppose that S is an ITTM-semi-decidable set, given by a program P. We define S_α as the set of z such that $P(z)$ halts before α. Note that if M is admissible and contains a code for α, then there is a Borel code for S_α in M and hence $\mu(S_\alpha)$ can be calculated in M. In particular, $\mu(S_\alpha)$ is ITTM-writable from any code for α. Moreover, α is ITTM-writable in z since $\alpha < \lambda^z$. Hence there is a code for α in L_{λ^z}. Let $\hat{S}$ be the set of all z such that there exists some $\alpha < \lambda^z$ with $z \in S_\alpha$ and $\mu(S_\alpha) = 0$. Moreover, let $\hat{S}_\alpha$ denote the set of z with $z \in S_\alpha$ and $\mu(S_\alpha) = 0$. Since the set of z with $\lambda^z = \lambda$ is co-null by Theorem 3.13, $\hat{S}$ is the union of a null set and the sets $\hat{S}_\alpha$ for all $\alpha < \lambda$. □

The universal test is the union of all sets $\hat{S}$, where S ranges over the ITTM-semidecidable sets. The following notion is analogous to Π^1_1-random.

The following is a variant of van Lambalgen's theorem for ITTMs. We say that reals x and y are *mutually random*, in any given notion of randomness, if their join $x \oplus y$ is random.

Lemma 4.3. A real x is ITTM-random and a real y is ITTM-random relative to x if and only if x and y are mutually ITTM-random.

Proof. Suppose that x is ITTM-random and y is ITTM-random relative to x. Moreover, suppose that x and y are not mutual ITTM-randoms. Then there is an ITTM-semidecidable null set A given by a program P such that $x \oplus y \in A$. Let $A_u = \{v \mid u \oplus v \in A\}$ denote the section of A at u. Let

$$A_{>q} := \{u \mid \mu(A_u) > q\}$$

for $q \in \mathbb{Q}$. Note that $u \in A_{>q}$ if and only if some condition in L_{Σ^u} with measure $r > q$ in $\mathbb{Q}$ forces that $P(\check{u}, \dot{v})$ halts, where $\dot{v}$ is a name for the random real over L_{Σ^u}, by Lemma 2.6. This is a Σ_1-statement in L_{Σ^u} and therefore in L_{λ^u}. Then the set $A_{>q}$ is semidecidable by Lemma 3.4, uniformly in $q \in \mathbb{Q}$. Since $\mu(A) = 0$, $\mu(A_{>0}) = 0$. Since x is ITTM-random, $x \notin A_{>0}$ and hence $\mu(A_x) = 0$. Note that A_x is semidecidable in x. Since y is ITTM-random relative to x, this implies $y \notin A_x$, contradicting the assumption that $x \oplus y \in A$.

Now suppose that x and y are mutually ITTM-random. To show that x is ITTM-random, suppose that A is a semidecidable null set with $x \in A$. Then $A \oplus {}^\omega 2$ is a semidecidable null set containing $x \oplus y$, contradicting the assumption that x and y are mutually ITTM-random. To show that y is ITTM-random relative to x, suppose that y is an element of a semidecidable null set A relative to x. Since the construction of $\hat{S}$ in Lemma 4.2 is effective, there is a semidecidable null subset B of ${}^\omega 2 \times {}^\omega 2$ with $A = B_x$ (in fact, all sections of B are null). Then $x \oplus y \in A$, contradicting the assumption that x and y are mutual ITTM-randoms. □

The next result is analogous to the statement that a real x is Π^1_1-random can be characterized by Δ^1_1-randomness and $\omega_1^x = \omega_1^{\mathrm{ck}}$ (see [Nie09, Theorem 9.3.9]). The fact that $\zeta^x = \zeta$ holds in the next result was noticed by Philip Welch.

Theorem 4.4. A real x is ITTM-random if and only if it is random over L_Σ and $\Sigma^x = \Sigma$. Moreover, this implies $\lambda^x = \lambda$ and $\zeta^x = \zeta$.

Proof. First suppose that x is ITTM-random. We first claim that x is random over L_Σ. Since every real in L_Σ is accidentally writable, we can enumerate all Borel codes in L_Σ for sets A with $\mu(A) = 0$ and test whether x is an element of A. Therefore the set of reals which are not random over L_Σ is an ITTM-semidecidable set with measure 0, and hence x is random over L_Σ. We now claim that $\Sigma^x = \Sigma$. Since $\Sigma^y = \Sigma$ holds for all sufficiently random reals by Lemma 3.8, the set A of reals y with $\Sigma^y > \Sigma$ has measure 0. Since the existence of Σ is a Σ_1-statement over L_{Σ^y}, the set A is semidecidable. Since x is ITTM-random, $x \notin A$ and hence $\Sigma^x = \Sigma$.

Second, suppose that x is random over L_Σ and $\Sigma^x = \Sigma$. Suppose that A is a semidecidable null set containing x given by a program P. Then $P(x)$ halts before $\lambda^x < \Sigma^x = \Sigma$ and hence some condition p forces over L_Σ that $P(x)$ halts, by Lemma 2.6. Then $\mu(A) > 0$, contradicting the assumption that A is null.

To show that $\lambda^x = \lambda$, note that $L_\lambda[x] \prec_{\Sigma_1} L_\Sigma[x] = L_{\Sigma^x}[x]$ by Lemma 3.8. Since λ^x is minimal with this property, $\lambda^x \leq \lambda$.

To show that $\zeta^x = \zeta$, note that we have $L_{\zeta^x}[x] \prec_{\Sigma_2} L_{\Sigma_x}[x] = L_\Sigma[x]$ and hence $L_{\zeta^x} \prec_{\Sigma_2} L_\Sigma$. Since L_ζ is the only proper Σ_2-elementary submodel of L_Σ, the claim follows. □

This shows that the level of randomness in the assumption of Lemma 3.8 can be improved to ITTM-random for $\alpha = \zeta$, $\beta = \Sigma$.

Following the proof of the previous theorem, Philip Welch showed that it is further equivalent to assume that x is random over L_λ and $\lambda^x = \lambda$, or that x is random over L_ζ and $\zeta^x = \zeta$. These equivalences follow immediately from the the following result.

Theorem 4.5. The following conditions are equivalent for a real x.

(a) x is ITTM-random.
(b) x is random over L_Σ and $(\lambda^x, \zeta^x, \Sigma^x) = (\lambda, \zeta, \Sigma)$.
(c) x is random over L_{λ^x}.

Proof. It follows from Theorem 4.4 that (a) implies (b) and this implies (c). The following argument by Philip Welch proves the implication from (c) to (a). If $\lambda^x > \Sigma$, then the conclusion follows from Lemma 3.8. We can hence assume that $\lambda^x \leq \Sigma$. Moreover, assume that x is random over L_{λ^x} and that x is in an ITTM-semidecidable set A that is given by a Σ_1-formula φ. Then $L_{\lambda^x}[x] \vDash \varphi(x)$. Hence some condition p forces $\varphi(x)$ over L_{λ^x}. For any $y \in [p]$, we have $L_{\lambda^y}[y] \prec_{\Sigma_1} L_{\Sigma^y}[y]$, $\lambda^x \leq \Sigma \leq \Sigma^y$ and $L_{\lambda^x}[y] \vDash \varphi(y)$. Hence $L_{\lambda^y}[y] \vDash \varphi(y)$ and thus $y \in A$. It follows that $\mu(A) > 0$ and x must be ITTM-random. □

We obtain the following variant of Theorem 3.13.

Theorem 4.6. If x is computable from both y and z and y is ITTM-random in z, then x is computable. In particular, this holds if y and z are mutual ITTM-randoms.

Proof. Suppose that $P(y) = Q(z) = x$. Then $A = \{u \mid P(u) = Q(z)\}$ is semidecidable in z. If $\mu(A) > 0$, then x is computable from all element of a set of positive measure and hence x is computable by Theorem 3.13. Suppose that $\mu(A) = 0$. Then $y \notin A$, since y is ITTM-random in z, contradicting the assumption that $y \in A$. □

4.2. **A decidable variant.** Martin-Löf suggested to study Δ^1_1-random reals. The following variant of ITTM-randomness is an analogue to Δ^1_1-randomness.

Definition 4.7. A real is ITTM-*decidable random* if it is not an element of any decidable null set.

We now give a characterization of this notion. We call a real co-ITTM-random if it avoids the complement of every semidecidable set of measure 1. The following result is analogous to the equivalence of Δ^1_1-random and Σ^1_1-random [CY15b, Exercise 14.2.1].

Theorem 4.8. The following properties are equivalent.

(a) x is co-ITTM-random.
(b) x is ITTM-decidable random.
(c) x is random over L_λ.

Proof. The first implication is clear. For the second implication, note that since every Borel set with a Borel code in L_λ is ITTM-decidable, every ITTM-decidable random real x is random over L_λ.

For the remaining implication, suppose that x is random over L_λ and P is a program that decides the complement of a null set A with $x \in A$. Suppose that $\dot{x}$ is the canonical name for the random real (note that this name is equal for randoms over arbitrary admissible sets). Relative to every random real y over $L_{\Sigma+1}$, A is definable over L_Σ, since $\Sigma^y = \Sigma$ by Theorem 3.8. Hence $y \notin A$ and $P(y)$ halts before $\lambda^y = \lambda$ for any such real. Therefore in L_Σ, there is some γ (namely λ) such that the Boolean value of the statement that $P(\dot{x})$ halts strictly before γ is equal to 1. The existence of such an ordinal γ is a Σ_1-statement, hence there is such an ordinal $\bar{\gamma} < \lambda$ such that the statement holds in L_λ for $\bar{\gamma}$, by Σ_1-reflection. Let A denote the Boolean value of the statement that $P(\dot{x})$ halts before $\bar{\gamma}$. Then A is a Borel set with a Borel code in L_λ and $\mu(A) = 1$. Therefore $x \in A$ and $P(x)$ halts before λ, contradicting the assumption that $x \in A$. □

Hence the distance between the analogues to Δ^1_1-random and Π^1_1-random is larger than for the original notions.

Lemma 4.9. There is no universal ITTM-decidable random test.

Proof. Suppose that A is a universal ITTM-decidable random test. In particular, the complement of A is ITTM-semidecidable. By the characterization of ITTM-semidecidable reals in Lemma 3.4 and [SS12, Corollary 8], ITTM-semidecidable uniformization holds.[4] Therefore, every semidecidable set, in particular the complement of A, has a recognizable element. This contradicts the assumption that A is a universal test. □

We call a program P *deciding* if $P(x)$ halts for every input x. The following is a version of van Lambalgen's theorem for ITTM-decidability.

Lemma 4.10. A real x is ITTM-decidable random and a real y is ITTM-decidable random relative to x if and only if $x \oplus y$ is ITTM-decidable random.

Proof. Suppose that $x \oplus y$ is ITTM-decidable random. The forward direction is a slight modification of the proof of von Lambalgen's theorem for ITTMs in Lemma 4.3, so we omit it. In the other direction, the only missing piece is the following claim.

Claim 4.11. Suppose that A is a decidable set given and $A_x = \{y \mid x \oplus y \in A\}$ is null. Then there is a decidable set B such that $A_x = B_x$ and all sections of B are null.

Proof. It was shown in the proof of Lemma 4.3 that the set

$$A_{>q} = \{u \mid \mu(A_u) > q\}$$

is semidecidable for all rationals q, uniformly in q, since the statement $u \in A_{>q}$ is Σ_1 over L_{Σ^u}. Since $L_{\lambda^u} \prec_{\Sigma_1} L_{\Sigma^u}$, this statement reflects to L_{λ^u}. Let

$$A_{\geq q} = \{u \mid \mu(A_u) \geq q\},$$

Then the statement $u \in A_{\geq q}$ is equivalent to $u \in A_{>r}$ for unboundedly many rationals $r < q$. Since λ^u is u-admissible, this is a Σ_1-statement in u over L_{λ^u}. Hence $A_{\geq q}$ is semidecidable, uniformly in q.

Therefore, if A is decidable, then $A_{>q}$ and $A_{\geq q}$ are semidecidable, uniformly in q. Using the fact that $A_{=0} = \{u \mid \mu(A_u) = 0\}$ is decidable, it is easy to define a decidable set B as in the claim. □

[4]The proof of [SS12, Corollary 8] is a variant of the proof of Π^1_1-uniformization.

This completes the proof of Lemma 4.10. □

Lemma 4.8 and 4.10 immediately imply that x and y are mutually random over L_λ if and only if x is random over L_λ and y is random over L_{λ^x}.

The following variant of Lemma 4.6 for reals computable from two mutually randoms can be shown for the following stronger reduction. A *safe* ITTM-*reduction* of a real x to a real y is a deciding ITTM (i.e. P halts on every input) with $P(x) = y$. We call reals x and y *mutually* ITTM-*decidable random* if $x \oplus y$ is ITTM-decidable random.

Lemma 4.12. If x is safely ITTM-reducible both to y and z, and y and z are mutually ITTM-decidable random, then x is ITTM-computable.

Proof. Suppose that P is a safe reduction of x to y and Q is a safe reduction of x to z. Since P is a safe reduction, the set $A = \{u \mid P(u) = Q(z)\}$ is ITTM-decidable relative to z. As $P(y) = x = Q(z)$, $y \in A$. Since y is ITTM-decidable random relative to z, A is not null. Then P computes x from all elements of a non-null Lebesgue measurable set, and hence x is computable by Theorem 3.13. □

Lemma 4.10 can be interpreted as the statement that x and y are mutually random (i.e. $x \oplus y$ is random) over L_λ if and only if x is random over L_λ and y is random over L_{λ^x}, by the relativized version of Lemma 4.8.

Intuitively, a random sequence should not be able to compute any non-computable sequence with special properties, such as recognizable sequences. The following result confirms this.

Lemma 4.13. Any recognizable real x that is computable from an ITTM-random real y is already computable.

Proof. Suppose that P recognizes x and $Q(y) = x$. Then the set

$$A = \{z \mid P^{Q(z)} = 1\}$$

is semi-decidable and contains y, where $Q(z)$ is the output of the computation Q with input z. Note that x is computable from every element of A via Q. If A is not null, then x is computable by Theorem 3.13. If A is null, this contradicts the assumption that y is ITTM-random and thus avoids A. □

Hence there are real numbers that are not computable from any ITTM-random real, and therefore there is no analogue for ITTM-randoms to the Kučera-Gács theorem (see [DH10, Theorem 8.3.2]).

Remark 4.14. The previous results and proofs relativize to reals. Moreover, they do not use any specific properties of Lebesgue measure and therefore hold for arbitrary measures ν with the property that the function that maps $s \in 2^{<\omega}$ to $\nu(N_s)$ is computable. Finally, most results in this section hold for genericity instead of randomness and for some other machine models, for instance ITRM-genericity [Car16].

4.3. **Comparison with a Martin-Löf type variant.** Hjorth and Nies introduced a Π^1_1-version of Martin-Löf randomness [HN07] and proved variants of the Levin-Schnorr theorem, the Kraft-Chaitin theorem and the coding theorem. In particular, they showed that Π^1_1-ML-randomness can be characterized by initial segment complexity. They further compared this notion with Π^1_1-randomness and observed that the latter is strictly stronger. It is therefore natural to consider an ITTM-variant of Martin-Löf randomness.

We first discuss analogues of the theorems of van Lambalgen and Levin–Schnorr for ITTM_{ML}-random reals. In the following discussion, we will refer to [HN07, Section 3] and [BGM17, Section 1.1, Section 3] and expect that the reader is familiar with the results and proofs there. Moreover, since the proofs mentioned below are minor modifications of the proofs in these papers without new ideas, we will only point out the differences to our setting.

Towards van Lambalgen's theorem for $\mathrm{ITTM}_{\mathrm{ML}}$-random reals, one defines a continuous relativization as in [BGM17, Section 1.1] as follows. For any functional $\Psi \subseteq 2^{<\omega} \times 2^{<\omega}$ and $x \in 2^\omega$, we let

$$\Psi^{(x)} = \bigcup \{t \mid \exists n < \omega \ (x{\restriction}n, t) \in \Psi\}.$$

Moreover, a subset A of ${}^\omega 2$ is called $\mathrm{ITTM}^{(x)}$-semidecidable if $A = \Psi^{(x)}$ for some ITTM-enumerable set Ψ. One then obtains the following result as in [BGM17, Section 3].

Lemma 4.15. A real $x \oplus y$ is $\mathrm{ITTM}_{\mathrm{ML}}$-random if and only if x is $\mathrm{ITTM}_{\mathrm{ML}}$-random and y is $\mathrm{ITTM}_{\mathrm{ML}}^{(x)}$-random.

The difference in the proof is that ω_1^{ck} is replaced with λ and the projectum function on ω_1^{ck} is replaced with a projectum function on λ, i.e. an injective function $p\colon \lambda \to \omega$ such that its graph is Σ_1-definable over L_λ. For instance, we may consider the function p which maps an ordinal $\alpha < \lambda$ to the least program that writes a code for α. Moreover, the need for a continuous relativization is discussed in detail in [BGM17].

Towards a version of the Levin-Schnorr theorem for ITTMs, a standard argument shows that there is an effective list $\langle M_d \mid d \in \omega \setminus \{0\}\rangle$ of all prefix-free ITTMs. Such a list can defined effectively by replacing each ITTM P by a prefix-free ITTM $\hat{P}$, by simulating P on all inputs with increasing length. Given such a list, we obtain a universal prefix-free ITTM U by defining $U(0^{d-1}1\sigma) = M_d(\sigma)$. The ITTM-version of the prefix-free Kolmogorov-Solomonoff complexity is defined as

$$K(x) = K_U(x) = \min\{|\sigma| \mid U(\sigma) = x\}.$$

The following analogue to the Levin-Schnorr theorem, which characterize randomness via incompressibility, is proved as in [HN07, Theorem 3.9], by replacing ω_1^{ck} with λ.

Theorem 4.16. The following properties are equivalent for infinite strings x.

(a) x is $\mathrm{ITTM}_{\mathrm{ML}}$-random.
(b) $\exists b \ \forall n \ K(x{\restriction}n) > n - b$.

We now compare the introduced randomness notion with Π^1_1-randomness. It is easy to see that there is an ITTM-writable Π^1_1-random real. For example, let x be the $<_L$-least real that is random over $L_{\omega_1^{\mathrm{ck}}+1}$. Since L_λ is admissible and ω_1^{ck} is countable in L_λ, $x \in L_\lambda$ and hence x is ITTM-writable. Moreover, x is Π^1_1-random by Lemma 2.11 and Lemma 2.13.

The next results show that $\mathrm{ITTM}_{\mathrm{ML}}$-random is strictly between Π^1_1-random and ITTM-random. For the next lemma, recall that a real $r \in \mathbb{R}$ is called *left*-Π^1_1 if the set $\{q \in \mathbb{Q} \mid q \leq r\}$ is Π^1_1. We give a short proof of this result for the benefit of the reader.

Lemma 4.17. (Tanaka, see [Kec73, Section 2.2 page 15]) The measure of Π^1_1 sets is uniformly left-Π^1_1.

Proof. Using the Gandy-Spector theorem and Sacks' theorem that the set of reals x with $\omega_1^x = \omega_1^{\mathrm{ck}}$ has full measure (see Lemma 2.11), we can associate to a given Π^1_1 set a sequence of length ω_1^{ck} of hyperarithmetic subsets, such that their union approximates the set up to measure 0. This shows that the measure is left-Π^1_1. Moreover, in the proof of the Gandy-Spector theorem (see [Hjo10, Theorem 5.5]) for a Π^1_1 set ${}^\omega 2 \setminus p[T]$, the Σ_1-formula states that T_x is well-founded, and hence the parameter in the formula is uniformly computable from T, and the assignment is uniform. $\square$

Lemma 4.18. Every ITTM-random real is $\mathrm{ITTM}_{\mathrm{ML}}$-random and every $\mathrm{ITTM}_{\mathrm{ML}}$-random real is Π^1_1-random.

Proof. The first implication is obvious. For the second implication, suppose that $A = p[T]$ is a Σ^1_1. Using Lemma 4.17, we can inductively build finitely splitting subtrees S_n of T with $\mu([T] \setminus [S_n]) \leq 2^{-n}$, uniformly in n. Moreover, this sequence can be written by an ITTM. □

5. Questions

We conclude with several open questions about the properties of randomness notions. The following question asks if a property of ML-random reals and Δ^1_1-random reals (see [CY15b, Theorem 14.1.10]) holds in this setting.

Question 5.1. Is ITTM_{ML}-randomness strictly stronger than randomness over L_λ?

The fact that ITTM_{ML}-randomness is strictly stronger than Π^1_1-randomness suggests an analogue for Σ_n-hypermachines [FW11].

Question 5.2. Is every ML-random real with respect to Σ_{n+1}-hypermachines already semidecidable random with respect to Σ_n-hypermachines?

Since the complexity of the set of Π^1_1-randoms is $\mathbf{\Pi^0_3}$ [Mon14, Corollary 27] and this is optimal (see [Mon14, Theorem 28] and [Yu11]), this suggests the following question.

Question 5.3. What is the Borel complexity of the set of ITTM-random reals?

The set NCR is defined as the set of reals that are not random with respect to any continuous measure. It is known that this set has different properties in the hyperarithmetic setting [CY15a] and for randomness over the constructible universe L [YZ17].

Question 5.4. Is there a concrete description of the set NCR, defined with respect to ITTM-randomness?

Moreover, it is open whether Theorem 4.6 fails for ITTM_{ML}-randomness. More precisely, we can ask for an analogue to the counterexample or ML-randomness (see [Nie09, Section 5.3]).

Question 5.5. Let Ω_0 and Ω_1 denote the halves of the ITTM-version of Chaitin's Ω (i.e. the halting probability for a universal prefix-free machine). Is some non-computable real computable from both Ω_0 and Ω_1?

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